THE LARGER HUMAN WORTH OF MATHEMATICS

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MATHEMATICAL thought has exercised over my spirit a fascination which is far-reaching in its effect upon my activity and happiness. It is not an easy matter to present the characteristics of this thought to one who has not been initiated into the remarkable secrets of the science; indeed the difficulty is so great that the task has seldom been attempted and not always with happy consequences. And yet I have felt that I could not fail of moderate success in this matter if I could reflect with any skill the enthusiasms of my own delight, since I believe that the fire of natural and spontaneous interest in one mind has the quality of producing a corresponding exaltation in another.

There are elements of mathematical thought which illuminate my spirit with a brilliant radiance whose after-images are pleasant to contemplate. Perhaps I can not bring to you now one of these moments of illumination of joy, for one can not produce them to order or easily recapture them but I can hope to present certain after-images which show some qualities of the original.

The impulse for the advancement of knowledge which is both most fundamental and most far-reaching in its practical and ideal effects is that which grows out of the pursuit of truth for truth's sake. What do we mean by this? What do we mean by mathematics for its own sake? It is clear that we do not intend to set up mathematics as a monster which must be worshipped, whom it is our duty to delight with the incense of human sacrifice. We mean rather to direct attention to the human values which are inherent in it apart from its use as a tool in any of the varied ways in which it may be so employed. Our purpose is to focus attention upon its primary values, those which it has in and of itself, those which are intimate to its own character and do not depend upon its uses outside of its own domain.

We are fundamentally so constituted that we delight in knowing for the sake of knowing. It is our most abstract and our most general motive in science. It actuates most powerfully our choicest spirits, moving them sometimes with a fervor akin to that of re-

ligion. A marvelous curiosity to know, insatiable and always demanding further satisfactions, creates a longing which can be relieved only by knowledge. It projects itself into the unknown and leads the researcher in ways yet untrodden to a goal which can not be foreseen. At the outer boundary line of knowledge, faint glimmerings may be detected in the darkness of ignorance beyond. What beckons us forth we do not know. Whether it can bring us any good we have no means of foretelling. It may lead us to a tragical something which will make it necessary for us, in much pain, to cast away some of our most cherished prejudices. But, whatever lies beyond in that which is concealed from our present vision,

> We work with this assurance clear,
> To cover up a truth for fear
> Can never be the wisest way;
> By every power of thoughtful mind
> We strive a proper means to find
> To bring it to the light of day.

Systematic and unsystematic thought. In its further reaches mathematics is perhaps the most abstruse of our mental disciplines; but in its first stages it is the simplest of those sciences which have attained permanence of result. Mathematics is the field of thought in which permanent progress is easiest. It has obtained this facility through abstraction. The problems of nature are complex beyond our ability to cope with or perceive. In the first attempt to make progress in the way of definite conquest, we must abstract from the complexity of the situation and attain to a new one relatively much simpler. In fact we may find it necessary to create a new situation having certain analogies with the actual one of nature but being so much simpler that we are able to grasp far more successfully the interrelations of its parts. It is precisely this procedure which has guided the development of mathematics.

It is not that the mathematician refuses to be interested in the immeasurable complexity of nature. It is rather that he seeks permanence of conquest, even though it be at the expense (in the first instance, at least) of a narrowed range of use. The way in which mathematics has interacted usefully with other elements in the progress of thought justifies her method of abstraction as profitable; it certainly conforms to the requirements of esthetic delight for the mathematician himself.

But the abstractions of mathematics leave us in a rarefied atmosphere too far removed from concrete experience to be a satisfactory resting-place for the mind of an inquisitive organism like

man. He seeks to get closer to concrete phenomena. But, unless he is content to deal in vague and uncertain generalities, he finds the complexity of nature far too great for him even though he has forged a mathematical tool to assist in his labors. He must still confine attention to certain groups of phenomena abstracted from their surroundings. He must try, so to speak, to lift them from the matrix of their environment. Thus we arrive at the exact sciences of natural phenomena, as, for instance, the science of mechanics, through the use of abstraction as a necessary preliminary to exact and permanent intellectual conquest.

But it is clear that we do not understand even these restricted ranges of phenomena which we have separated in thought but not in reality from their environment until we have considered all the elements in that environment and have synthesized the disjoined knowledge of its parts into a comprehensive understanding of the whole. When we come to these questions of greater complexity we feel less certain of our results and far less confident of the permanence of our conclusions. The history of philosophy with its changing systems and the flux of its emphasis is a striking commentary on the difficulty of the general problem. And even here in the comprehensive problems of philosophy itself large abstraction has already been made from the complexity of phenomena and life and existence.

In definite contrast with the systematic thought of mathematics, the natural sciences, and certain parts at least of philosophical truth and speculation, stands the unsystematic thought of art and literature. Here one deals with the actual complexity of life and even with the character of individuals and their emotions. "It is the privilege of art to represent at a glance the whole of its object, and thus to produce at once a total effect on the mind of the beholder." Not infrequently men of science have seemed to overlook the importance of this body of unsystematic thought in art and especially in literature. But it appears that the development of unsystematic thought is necessary to sanity; not that its unsystematic character as such contributes to this end, but that through no efforts being made at systematic statement one can allow the whole flux of life to be reflected at once, at least so far as to have no purposed exclusions. If one is to have the systematic exactness of pure science it can be only after many relevant considerations are shorn off and attention is fixed on a part only (usually a small part) of the whole. This is necessary to definite conquest and the method is to be freely used. If one stops with this, however, a one-sided unbalanced view results which contributes forcibly to a lack of sanity in outlook and general judg-

ment. With the continued development of systematic thought, let us encourage and support the free development of unsystematic thought in poetry and other forms of literature and art. The latter have abiding qualities of intrinsic human worth which science can never replace.

The domains of systematic and unsystematic thought have usually little effect the one upon the other; and yet between these two great arteries of our culture there must be somewhere a vital connection. The historians of general thought have not yet properly taken into account the vast body of unsystematic thought in the literatures of the world where for millenniums it has awaited their research. Nor on the other hand has the poetry of exact truth been written nor have its cultural elements in any representative case found their way into the general thought of mankind.

Literature not only takes the complex whole of life at a glance but it also internationalizes its local subjects and gives them a value which endures independently of time and place. Mathematics and poetry lie, if not on, at least not far from the extremes, the one of systematic and the other of unsystematic thought, and thus are about as far removed as possible one from the other. And yet they have a very striking common property, namely, the property of permanence. No other large domains of thought than mathematics and literature have acquired large bodies of truth retaining their values essentially unimpaired for two thousand years, not in a stagnant state, but in a state of vitality and effectiveness. It is a matter of great inspiration to see the Greek geometry and the Greek tragedy surviving through the ages and retaining the active power to excite our admiration and increase our happiness to-day.

The language of exact truth. Communication to others requires the previous construction of a language having the requisite flexibility. If we look into the remote origins of our culture we shall find reason for believing that it was language which initiated the marvelous release of the powers of man inherent and undeveloped in our primitive ancestors and coming to their fruition only after many ages of progress. It was and is a fundamental element in accumulating and retaining the heritage of the past so that each new generation, in periods of development, is able to begin a little further on than the preceding one.

There is something subtle in the way in which language makes it possible to pass the experience and thought of one generation along to the next. The phenomena of nature present themselves to us ordered in space and time, but without apparent logical connec-

tions to bind them together. As long as we meet them merely in the multiplicity of their separate existences we can not get far towards an understanding of them or of a mastery over them. It is necessary that they shall be ordered into groups or sets, each held together by some tie which serves in our mind either as a unifying or as a connecting element. The combination of distinct elements into a whole and the formation of these groups depends on a process which the mind constructs for itself slowly and only after much labor. Any means of giving a considerable measure of permanence to the constructions of one individual mind or of one age, will be of great value in maintaining mastery and effecting its further development.

Let us conceive, if possible, the condition of prehistoric man at a time when language was in the process of construction for the first time. When a tribe of men reach agreement concerning the common elements of a set of objects, as for instance the trees of the forest, and signalize a realization of their common features by giving to them some such name as tree, they crystallize into definite form a class of experiences felt by each of them in a more or less vague way. The idea denoted by the word becomes more distinct by constant recurrence and both word and idea take their places as part of the mental possessions of man.

This primitive process has been repeated in all ages of our history; it recurs often in the present day, notably in connection with the development of scientific thought. In youth we listen to the words of those of the previous generation, trace in their features some mark of the anxieties through which they have lived, and share remotely their enthusiasms and aspirations, their passions and their joys. But we receive through them in the language which they teach us a more living inheritance and a more eloquent testimonial to their ways of thought. "Unknowingly they have themselves altered the tongue, the words and sentences, which they received, depositing in these altered words and modes of speech the spirit, the ideas, the thought of their lifetime. These words and modes of speech they handed down to us in our infancy, as the mould wherein to shape our minds, . . . as the instrument with which to convey our ideas. In their language, in the phrases and catchwords peculiar to them, we learnt to distinguish what was important and interesting from what was trivial or indifferent, the subjects which should occupy our thoughts, the aims we should follow, the principles and methods which we should make use of."

A word or a way of thought into which so much experience of the race has been instilled can easily be taught to the children of

a new generation and be made to serve for them as a nucleus about which they can gather experiences of their own similar to those first embodied in the language. Thus through the various words which they use and the various turns of phrase which they employ they have a subtle means of assistance in organizing their early experience so that they are able to make much more rapid acquisition of knowledge than their ancestors who first had the confusion of unorganized impressions out of which to construct the initial organization of truth.

To the individual who is brought up in a civilized and intellectual age words and their organization into sentences certainly come earlier than clear and conscious thought. Through the use of our parents' tongue we are introduced to the complex processes of highly abstract reasoning in a manner which is truly marvelous. The way of thinking of our ancestors, preserved in some measure in the constructions of their language and in the peculiar ways of expressing thought developed through ages of progress, becomes to us our most precious heritage from the past. A highly significant part of the development of mankind is summarized into the forms and words of language in such a way as to be capable of transmission and to be of unmeasured value in passing on to the children the acquisitions of their ancestors.

What the language of daily life does for the thought of usual intercourse the language of mathematics does for the thought of exact truth. Everything which I have said about language in general I can now transfer to the language of mathematics in respect to its use in connection with exact thought. It furnishes the essential means for the expression of the latter. It supplies the support without which the mind would be unable to carry through the processes necessary to attain the more profound or far-reaching results. There is a certain storing, as it were, of intellectual force in the mathematical symbols from which it can be released suddenly with almost explosive power. These become mighty engines through the aid of which we can rear intellectual structures quite inaccessible to our unsupported power.

The invention of number was the first step in the creation of the language of mathematics; and the choice of adequate and convenient symbols for the representation of integers is one of the chief triumphs of the intellect. A long and arduous mental struggle, in which some of the finest minds of antiquity had part, preceded the conception of zero and the introduction of a symbol to represent it in the way now familiar even to our children. The result of a long and important development of thought is embodied compactly in this remarkable sign. The introduction of a symbol

like 4/5 marks a new stage in the development of mathematics. The general fact is repeated in many situations; but I can not go into a further analysis of this matter. It is sufficient to our purpose if we realize that the language of mathematics is an essential support to the mind in all its processes of exact thought and that the results emerging in this way can be expressed only in mathematical terms.

Being a lover both of mathematics and of poetry I enjoy finding certain general similarities between them. I have already alluded to one very striking common property of them, namely the property of permanence. I wish now to direct your attention to the historical fact that poetry was the primary and most important means by which the language of ordinary intercourse was brought to a stage of relative perfection just as mathematics was the essential means in creating the language of exact truth. Ordinary language having been brought to perfection by the labors of the poets was then appropriated by writers of prose; exact language having been developed by the mathematicians has been employed freely by the cultivators of every exact science.

Mathematics and philosophy. Philosophy and mathematics started life together. After a brief period of companionship they parted company and each went its own way. Mathematics was the first science to emancipate itself from the tutelage of philosophy; it gained its freedom at the dawn of Greek civilization. Mechanics next succeeded towards the close of the Grecian period, physics obtained its independent position at the opening of the modern era, biology about the beginning of the nineteenth century, and psychology in its latter half. Sociology as an independent science has hardly yet passed its period of infancy.

When men began consciously to cast about them to understand their universe, they found it possible in a relatively short time to procure and contemplate a large body of unsystematic thought, a wealth of philosophic explanation, and a rather large body of speculative proto-science of nature. But in respect of mathematical knowledge they had to begin much nearer the bottom. In the less exact disciplines there was an ebb and flow of movement with a general progress forward, accompanied often by a discarding of what at one time was considered well established. But in mathematics a conquest once made is almost never lost and there is a consequent unbroken enlargement of doctrine. Since it pushes its conquests out in many directions, is frequently annexing new domains, never yields up what it has once attained, and remains youthful in its spirit of conquest, mathematics is destined to become, if indeed it is not already, the most extensive scientific doctrine in the whole range of knowledge.

Early in the history of thought philosophy soared the heights on wings of speculative grandeur and soon reached an eminence which it has never surpassed. Mathematics took time to dig deep till it was in possession of secure foundations on which to build. Here it reared a magnificent structure of enduring beauty. In our generation this mathematics has reached forth a hand of conquest and has annexed certain restricted domains of philosophy. "The first real advance in logic since the time of the Greeks was made independently by Peano and Frege—both mathematicians" working with the tools and from the point of view of mathematics. In former days the nature of infinity and continuity belonged to philosophy, but now it belong to mathematics. An important part of the theory of classes has been annexed by this greedy conqueror.

But more than all this, it has injected its spirit into a large province of modern philosophy. Among the philosophies of the present day Bertrand Russell distinguishes three principal types, combined in varying proportions in single philosophers but in essence and tendency distinct, namely, the classical tradition, evolutionism, and the method of logical atomism. The last has crept into philosophy through the critical scrutiny of mathematics. According to Russell it represents "the same kind of advance as was introduced into physics by Galileo: the substitution of piecemeal, detailed, and verifiable results for large untested generalities recommended only by a certain appeal to the imagination."

A doctrine which lay quiescent in the domain of philosophy for many generations has recently been brought by mathematical methods into the activity of vigorous life. The modern theory of relativity is a precise physico-mathematical realization of the philosopher's speculation of relativity. The existence of the philosophical doctrine has been of profound value in the creation of the mathematical doctrine; but the latter is now so far in advance of the former that the philosopher is rare who is able to follow the train of thought by which the more exact theory is brought to fullness. This mathematical conquest of a domain of philosophy has in our generation yielded a penetrating insight into certain fundamental matters both of physics and of philosophy. A theory of gravitation, satisfying for the most part in its broad aspects, has come into being for the first time. Under the impulse of this theory our notions of force and mass have suffered considerable change and our conceptions of time and space have undergone a veritable revolution.

Without going into more detail in these matters, I may insist upon the fact that one of the profound intrinsic human worths of mathematics lies in its conquest over intellectual matters of peren-

nial interest on which agreement cannot be reached until they are penetrated by the spirit and methods of mathematics and the invariant elements of truth are extracted and justified by a convincing array of precise evidence.

Mathematics and the foundations of science. Mathematics is autonomous. What is intimate to it, its nature and structure and laws of being, must be sought in itself. Logically the mathematical sciences can be developed in complete independence of all other sciences; and when pursued in this way to their goal they completely realize their object. Owing to its self-sufficiency, its abstract character, and its exclusion of complicating factors from the ideal considerations with which it is concerned, mathematics is essentially easier than the other branches of systematic truth. The appearance of greater difficulty, which has deceived most people, grows out of the fact that it is relatively further advanced than any other subject. It requires the learner longer in mathematics than in other sciences to attain an elevation from which he may enjoy the prospect of unexplored territory. Its wildernesses are further from the confines of civilization; and the ignorant picture them as filled with horrid monsters of indescribable physiognomy. But the hardy intellectual traveler who explores in this land of the far unknown finds nature gracious, there as well as here, in dispensing her beauties and joys and comforts.

As a discipline which is unique through its being more completely developed than any other it may be utilized as an object lesson of importance in the development of thought. The mind has not been able to chart unknown regions and to explore them systematically. Truth, when attained, often has an appearance quite unexpected. Its central characteristics can not be anticipated before it appears in thought. Consequently, the extended development of any discipline affords a means of analyzing the methods and foundations of successful thinking and of extracting by such analysis principles of guidance for all domains of exact thought. In certain important respects mathematics affords just such a support to the mind in finding its way to truth; it has continuously rendered a service of this character since the days of the early Greek philosophers. This contribution has varied in detail from age to age, ranging from marvelous uses in interpreting physical phenomena to marked support in speculative philosophy and the theory of knowledge.

During the last half century or so mathematics has come definitely to a stage of self-consciousness with respect to its processes and presuppositions; and these have been analyzed and subjected to critical logical scrutiny. The foundations on which the

subject is built are understood with a completeness foreign to any other domain of thought. From this fact it may well serve as a matter of instruction to point the way to a suitable and needed analysis of the presuppositions on which any given discipline is founded.

The importance of such an analysis seems not always to be apprehended. The sciences of nature are shot through with presuppositions not recognized. Even in the more precise reasoning of mathematics there was much to be elucidated by a critical scrutiny; and certain presuppositions had to be brought into the focus of attention before it could be properly said that we understood the foundations of the science. Elsewhere such analysis has been made only very imperfectly; the success achieved by mathematics in this work has not yet borne its proper fruit. It has been made clear that no science has been brought to a truly objective stage in its development until the presuppositions lying back of it are perceived as such and the grounds for making them are clearly realized. In the sciences of nature this process is more difficult than in mathematics; this, indeed, accounts for the fact of earlier success in mathematics than elsewhere. But when the result is once achieved in one science no other should rest satisfied until the same end is reached in an appropriate way.

Mathematics and the method of thought. Perhaps it will be agreed that we can nowhere study the processes of thought, by which the intellect reaches appropriate decisions, more effectively than in that domain where it has been most successful in attaining enduring results of significant value. If this principle is agreed upon, it is a corollary that mathematics is a field of thought which will yield us some of our most definite information as to the essential elements in the methods of clear and accurate thinking. Unfortunately, mathematicians themselves have generally been but little interested in the broad principles of method which their achievements are capable of bringing to light; they have usually been disposed to stand apart from the broader questions of a theory of knowledge, satisfied apparently with the self-sufficiency of their own discipline. Outside of their fold there has never been a group of thinkers with the requisite information and training to elicit from the body of mathematics the instruction which it is thus capable of affording. This field of promising possibility lies uncultivated while we lack those advantages which its fruitage well may yield.

It is important to ascertain the character of those regions of thought in which new methods have most frequently arisen into clear consciousness. Owing to precision of ideas and processes

in mathematics we can answer that question definitely and with considerable confidence for that discipline. New methods have usually come to light in connection with well-defined and well-restricted problems. Experience has forced upon us a realization of the profound importance of deep penetration into even the simplest matters. When a new means of illuminating them has been discovered its radiance spreads to adjacent fields and often overleaps great barriers to shed new light in most unexpected places. The connections between different elements of thought can not be anticipated successfully; it requires the event to exhibit them. The presuppositions which underlie truth become apparent gradually as we derive the remotest consequences of what is already known. For the researcher everywhere the character of the success in mathematics emphasizes the importance of detailed and penetrating and carefully analyzed investigations of basic matters.

The continuous advance in the understanding of the presuppositions of our science, the axioms or postulates on which it rests, and the resulting modifications in our views of its significance impress us constantly with the supreme necessity of the logical coherence of knowledge. No principle is thoroughly understood until all of its consequences are developed and their ramifications are ascertained. This process can be carried out only through the most searching logical scrutiny; it is desirable that the intuition shall be present in discovery, but the logical faculties should dominate exposition completely. "To supersede the employment of common reason, or to subject it to the rigor of technical forms, would be the last desire of one who knows the value of that intellectual toil and warfare which imparts to the mind an athletic vigor, and teaches it to contend with difficulties and to rely upon itself in emergencies." But when its results are once attained and they are to be put to the test of a systematic organization for determining the coherence and consistency of the parts, no glow should be permitted except that which comes from the cold light of logic.

A more deep-lying problem of the method of exact thought is brought out by the question as to the fundamental character of that mental process by which scientific truth is discovered. Natural science always proceeds in one of three ways: mathematically, experimentally, or by hypothesis. Have all these methods fundamental matters in common one with another and with the processes of mathematics itself? And, if so, are they of such sort that it is useful to the progress of science or to our delight in it to have them brought to attention? Owing again to the relatively more advanced state of mathematics as compared with the natural sciences we can consider for it, in a more objective way than for them, this

question as to the basic characteristics of the process of discovery.

Not a few mathematicians are agreed that these characteristics are summed up in considerable measure in the word invention. Some of the things in mathematics one may think of as being discovered; but others, and the more fundamental things, seem to have been created by the mind. The positive integers, for instance, were not found in nature but were created by the human spirit. After their creation many of their properties have been discovered. This relation between invention and discovery pervades most of the mathematical literature. Mathematical space has been created, not found in nature, as is shown by the fact that the mathematician has several kinds of three-dimensional space as well as numerous spaces of higher dimensions. It is true that his creative power was released through observation of the environment; but it can not be maintained that the environment dictated the geometry since in that case only one geometry could have resulted. A full analysis of the matter would carry us much too far afield, but we may assert that the process of discovery in mathematics is primarily that of invention.

This leads the mathematician to suspect that the method of exact thought everywhere is largely dependent upon invention, that the hypotheses of science are not extracted from nature but are invented by the mind through a release of its powers brought about by natural phenomena. Since one's procedure in forming hypotheses is doubtless much affected by his conception of the nature of the process it is important that the laborers in each science shall ascertain the corresponding fundamental characteristics of their processes of discovery.

If we should suppose that the advance of knowledge among the most cultivated people is in the direction of making life not worth while this would operate to destroy the part of society so affected with pessimism and the whole earth would ultimately be left to the less advanced. Thus a philosophy which makes life not worth while will have a natural tendency to destroy itself, so that it can not become permanent. That philosophy of the method of thought which results from a contemplation of mathematical progress leads to a doctrine which dignifies the process of thinking, exalting it to a place of veritable creative grandeur. It proceeds in the direction of making life worth living. It is optimistic in outlook and thus has one of the first qualities which are essential to permanence.

The invariants of human nature. Another value of mathematics is in its creation or clarification for its own use of various concepts which are afterward seen to serve as a unifying element

about which other large domains of truth may be systematically organized and the relations of fact thus be brought to clearer understanding. Everywhere we are confronted by change; nature seems to be in an eternal flux. The complexity of particular phenomena is bewildering and we should be lost in their maze if we could not find some means of ascertaining the elements of permanence in the midst of the flux. In mathematics we have the same situation freed of distracting elements and idealized in a way which makes it possible to give a rather complete analysis of the whole matter. The flux and change of nature is replaced, in the ideal situations of mathematics, by what we call a group of transformations. The elements in consideration are subject to transformation according to the laws prescribed by the group which governs the phenomena; and our problem is to determine the things which are unchanged in the midst of the general flux allowed by the controlling group; in other words, we are seeking what we call the invariants, or the invariant combinations, of the elements subject to the flux permitted by the group. This conception, vaguely present in much of scientific speculation, has been recreated by mathematics into precise form, has been clarified, and has been utilized so fully that we now find it to be true that a large portion of the whole of mathematics has to do consciously or unconsciously with the theory of invariants. The essential elements of the logical characteristics of a situation of this sort are brought out clearly by the mathematical theory. The resulting body of truth furnishes us with a model by which we may be guided in the contemplation of the elements of permanence in any changing situation.

Whatever the subject of inquiry in any domain of exact thought there are certain entities whose mutual relations we desire to ascertain. The combinations which have an unalterable value under the changes to which the entities are subjected are their invariants. It is the purpose of the theory of invariants to determine these combinations, elucidate their properties, and express in terms of them the laws which are involved in the given situation. The "laws of nature" are expressions of invariant relations under the changes occurring in nature or brought about by directive agency. Two problems concerning natural phenomena demand attention. If we know the group of changes we may demand the determination of the invariants; if we know the invariants we may demand the determination of certain (if not all) possible groups under which these invariants persist. To enforce the judgment that invariants are a fundamental guide in present day science we have only to cite the fact that the theory of relativity has been devel-

oped in intimate dependence upon and under the guidance of the theory of invariants.

To pursue this matter further would carry us too far in the direction of a study of the usefulness of mathematics in the development of natural science, a matter which we are purposely excluding from present consideration. It has been said by them of old time that the proper study of mankind is man. Our purpose keeps us closer to this problem than to the study of nature. It is a fair question to ask what mathematics has to teach us concerning human nature. What do we mean by human nature except those characteristics of individual people which are unchanged from one to another, and from age to age? Those elements which are invariant through the whole group of human beings as far as they may be brought under observation? And how shall we determine the characteristics of this human nature other than by an analysis of the invariant elements in human experience and thought?

It can not be maintained that mathematics affords the best means for pursuing this study. In fact, it is probably generally supposed that mathematics makes no contribution at all to the problem of human nature, of the invariants among the qualities of individuals; we shall attempt to show that this judgment is incorrect.

The best means of studying human nature of course arises from the usual relations of life. But these in themselves are quite insufficient for a complete analysis. The continued acceptance of a large body of vital mathematical truth through some millenniums suggests the invariant character of certain elements of human thought in its logical aspects, just as the continued appeal of ancient poetry (for instance) to people of cultivated taste bespeaks an unchanging element of human nature in its finer emotional aspects. The presence of such invariant elements, wherever they may be found, is an instructive matter for the historian of culture and civilization.

Where can one find a systematic analysis of literature, that great storehouse of material for the understanding of human nature and the progress of unsystematic thought, having for one of its primary objects the ascertainment of the invariant elements of human nature in its emotional aspects? A study of the changes in taste and their cause contributes indirectly to this end; but both literature and mathematics, in different ways and with reference to different parts of our nature, can be made to yield important values towards an understanding of its invariant elements.

Since the historian of thought and civilization is seeking to bring his analysis of the progress of culture into systematic form

it is perhaps no great surprise that he has found it difficult, and so far has not found it possible, to utilize successfully the truth which is half-concealed and half-revealed in the unsystematic thought of literature. But it is rather astonishing that such historians of thought have not been able to utilize the systematic work of mathematics in their expositions. I know of only a single instance where a general analysis of the progress of thought has taken an adequate account of the domain of mathematics, and that is in the work of J. T. Merz on "The History of European Thought in the Nineteenth Century."[1] This excellent general analysis has not had for one of its purposes to bring out the invariant qualities of human thought, and hence of human nature, as they are made manifest by the abiding truths of mathematics. The contribution which mathematics has to make to the study of human nature has not yet been considered in a systematic way.

And still it is certain to those who contemplate the nature of mathematical truth that many characteristic qualities of human thought are to be determined from such an analysis. It is a significant fact for the understanding of ourselves that the demonstrations in Euclid's *Elements* gain the same adherence to-day as in his time and in all the intervening ages; an invariant quality in the processes of reasoning and the ground for conviction through demonstration abides through the ages. There is absolute agreement in all times and all places that the number of prime integers is infinite, bespeaking a unity of the whole race in its understanding of the properties of elements conceived in the first place with exactness. The properties of a Euclidean triangle are in harmonious agreement even though they have been discovered by numerous thinkers of many generations. A sphere did nothing for the Greeks contrary to what it does for us to-day. The properties of a cube are invariant, whoever derives them and in whatever age he lives. It is an eternal truth that every integer is the sum of squares of four integers, and there is unanimity as to this fact and as to its demonstration. The persistence of mathematical theorems and the continued agreement as to their proof indicates a profound unity in the characteristic thought processes of those who contemplate them, exhibiting one fundamental phase of human nature.

Artistic delight in mathematical truth. Truth serves many ends. When a science has reached a certain stage of development,

[1] To this magistral work and to many articles in the *Encyclopædia Brittannica* I am under deep obligations in connection with this address. I have also profited by reading C. J. Keyser's *The Human Worth of Rigorous Thinking*, Columbia University Press, 1916.

varying greatly with the character of its material, it begins to throw off into the body of society great practical or even esthetic values which could not be realized without it. Astronomy has enabled us to have some conception of the vastness of space and the hugeness of the mass of matter, perhaps infinite in its totality, distributed through this space. Geology has released the imagination to contemplate enormous periods of time and, through its influence on biology, has rendered marked service in making possible our conception of the long progress of life on the planet, culminating in man. Mathematics, by exhibiting a body of truth which can live through millenniums without needed corrections, and at the same time can grow in magnitude and range and interest, has given the human spirit new ground for believing in itself and for rejoicing in its power of consistent thought.

It is not enough to accumulate the elements of knowledge or even by means of them to control nature for our use; we must appropriate them by idealizing them into things of beauty and motives to conduct. Truth may be made to yield the highest delights of contemplation in the spirit of artistic performance. This is generally realized in the case of the unsystematic truth afforded by literature and the other fine arts. It is less in evidence in the greater body of systematic truth. But when the latter is brought to its highest order of perfection, as it is in the domain of mathematics, it becomes capable of yielding the purest and most intense delights in artistic excellence. They are of a sort to be enjoyed in large measure only after an adequate training; and in that respect there is a certain exclusiveness about them. But to those who are willing to pay the price of adequate knowledge mathematics yields a gratification of the artistic sense surpassed by that arising from no other source. "The musician plays and the artist paints simply for the pure love of creation." The mathematician creates abstract and ideal truth for the pure love of discovery and of contemplation of the beauty of his mental handiwork.

In pursuing esthetic satisfactions we create a beautiful theory for the sake of our delight in it, as in the case of the theory of numbers or of abstract groups. Working in such fields with the simpler elements of mathematical thought we make progress of a sort not at first possible with the more complex materials. We bring the theory to a higher state of perfection; there are fewer lacunæ; the connections of the various parts are exhibited with clarity; we have a sense of having seen to the root of the matter and having understood it in its basic characteristics. The theory thus developed becomes an ideal in the light of which we get a new

conception of what should be attained in other fields where the labor and the difficulty are greater. Results in one field of mathematics may thus become of great value in a totally different range of mathematical ideas or even in other disciplines altogether. Moreover, when such progress is attained we often find that the tools employed in bringing it about are sufficient for dealing with more difficult matters, so that the one completed theory furnishes us not only the ideal, but also the means, for further valuable progress.

A characteristic delight in mathematical truth is that which arises from economy of thought realized through the creation of general theories. When we develop the consequences of a set of broad hypotheses we find that our results, which are attained by a single effort, have applications at once in many directions. Thus we see the common elements of diverse matters and are able to contemplate them as parts of a single general theory pleasing for its elegance and comprehensiveness.

Fundamentally mathematics is a free science. The range of its possible topics appears to be unlimited; and the choice from these of those actually to be studied depends solely on considerations of interest and beauty. It is true that interest has often been, and is to-day as much as ever, prompted in a considerable measure by the problems actually arising in natural science, and to the latter mathematics owes a debt to be paid only by essential contributions to the interpretation of phenomena. But, after all, the fundamental motive to its activity is in itself and must remain there if its progress is to continue.

"The desire for the one just form which always inspires the literary artist visits most men sometimes" and is ever present to the mathematician in his hours of creative activity. The one just form which the mathematician seeks is more ideal and perhaps more delightfully artistic than that sought by any other thinker; for it is primarily a form of abstract thought in which he is interested, a form which remains the same as ages come and go, as languages are developed and die away, as the canvas of the painter rots to fragments and the material of the sculptured image is resolved by decomposition into its elemental dust. It is a thing of beauty which is indeed a joy forever.

For many people the numerous practical applications of mathematics have obscured its artistic elegance. But it is not the only fine art which in another aspect is also of the greatest practical utility. This quality it shares with the noble art of architecture. The two equally satisfy the following informal definition given

by Sidney Colvin: "The fine arts are those among the arts of man which spring from his impulse to do or make certain things in certain ways for the sake, first, of a special kind of pleasure, independent of direct utility, which it gives him so to do or make them, and next for the sake of the kindred pleasure which he derives from witnessing or contemplating them when they are so done or made by others." Both mathematics and architecture possess all the qualities here enumerated. Each is of essential practical utility, contributing necessary elements to the material comforts of man. And, more than this, each delights the artistic sense through beauties peculiar to itself and furnishing the ideal reason for its existence.

From a certain point of view the four main divisions of thought—mathematics, natural science, philosophy, that unnamed one ruling without definite system in the domain of art and literature—are the stones and brick and mortar from which is builded the culture of the time, into which are wrought the values received from the past, and through which our development shall proceed to the acquisition of new power for further conquest. We break the environment into parts in thought and from these we fashion new objects such as never before existed in the universe—objects both concrete and ideal—and these we put together in ways well pleasing to ourselves to serve the ends we propose or erect the constructs we conceive.

But this is too mechanical to be the whole truth. The more profound values lie deeper and have their fruition only in the fullness of the character of man. If science did not touch a more profound matter than mere motion or reach to constructs which can not be adequately pictured by material symbols, it would fall far short of the glory of Living Thought. But it does forcibly react in a profound way with all our activities, particularly through the emotions excited by the play of the artistic sense. In fact, the elements of all Thought are parts of one body, living and organized, inspired by the breath of the Universe itself and pulsating with the life of truth in its deeper manifestations.

The problem of consistent thinking. The leading characteristic of man is the power to think. There is nothing of higher esthetic interest than to determine whether we can think consistently. This fundamental question can be answered in the affirmative only by exhibiting the results of consistent thinking. The existence of mathematics affords the best conceivable proof of its possibility and gives the spirit of man leave to believe in itself, since here admittedly is a body of consistent thought maintaining

itself for generations and even for millenniums, able to sustain all the attacks of logic and all the tests of the practical life.

There was a time when this confidence in the permanence and consistency of mathematics was absolute. The fundamental methods of argumentation men conceived to belong to a class of innate or inherent ideas which had been put in the mind of man by the Creator. The initial hypotheses and basic notions of a mathematical discipline they thought of as belonging to the same category. If these innate ideas did not have all the elements of absolute certainty, there could be only one conclusion: the Creator had deliberately deceived man. Since they considered this to be absolutely impossible, they had complete confidence in the certainty of mathematical results.

Nowadays we seek a more earthly reason for confidence in our constructions in science. Our agreement that mathematics is possible as a consistent body of truth we now understand to rest on postulates for which admittedly we have no logical demonstration. Perhaps these postulates may be framed in the following way: Reasoning is possible and does not lead to wrong results when employed according to the universally accepted rules; mathematical objects can be created or discovered by the mind; we can actually formulate consistent axioms or postulates concerning these objects. With each of these three statements there are grave logical difficulties. We can not assert that we have an immediate perception of them as true; we can not, by direct illumination, see their validity. We must examine each of them in the cold light of experience and accept it only in so far as it meets the most exacting demands. Our confidence can never be absolute. As J. B. Shaw has said in his "Philosophy of Mathematics:" "We may found our deductions on what premises we please, use whatever rules of logic we fancy, and can only know that we have played a fruitless game when the whole system collapses—and there is no certainty that any system will not some day collapse!"

Let us proceed further with the difficulties of the situation. In all preceding generations conceptions in mathematics have been used with confidence which, in the experience of a later day, were found to be not sufficiently well defined; they have been discarded or essentially modified, sometimes after generations of confident use. It is not likely that men have heretofore always made mistakes of this kind and that we have suddenly come upon an age in which mathematical conceptions are refined to the last point of analysis.

We are then forced to the conclusion, however unwelcome it may be, that the certainty of mathematics after all is not absolute,

but relative. To be sure, it is the most profound certainty which the mind has been able to achieve in any of its processes; but it is not absolute. The mathematician starts from exact data; he reasons by methods which have never been known to lead to error; and his conclusions are necessary in the sense, and only in the sense, that no one now living can point to a flaw in the processes by which he has derived them.

Let us make as concrete as possible the difficulties and the immense values which are at stake. Let us suppose that the Euclidian geometry should become untenable under the weight of constant accretions and should go crashing down to helpless ruin; in that day man's hope of reaching tolerable certainty anywhere in his thinking would be destroyed and even the world of mind would become a dark confusion of irrational elements. The character of such a loss suggests the magnitude of the present value.

What certainty have we that such loss does not impend? We have no logical demonstration of its impossibility; and in the nature of things can not have such a demonstration. The same uncertainty attends all other truth, and in even more marked degree. There is no logical certainty of the consistency or the permanence of truth; at most there is a moral certainty. From mathematics we have the strongest grounds for the latter. When thousands of persons through thousands of years examine thousands of theorems proved by numerous methods and in numerous connections and there is always absolute unanimity in the compelling character of the demonstration and the consistency of results, we have a ground of moral confidence so great that we can dispense with the proof of logical certainty and comfortably lay out our lives on the hypothesis of the permanence, consistency and accuracy of mathematical truth. The existence of mathematics gives the mind the best reason yet advanced for believing in its powers and the essential accuracy of its careful processes.

Emotional exaltation arising from the contemplation of mathematical truth. A profound emotional exaltation arises from the contemplation of mathematical truth either in the static aspect of accomplished results or in the dynamic aspect of a science with an everlasting urge to further development. By the ideal values which it constructs and by the permanence of its results mathematics gives to the spirit of man the right and the courage to believe in itself and to trust its controlled flights. Here it justifies its claims to preeminence more completely and more profoundly than in any other part of its broad domain. It exhibits a body of truth which is permanently pleasing and which exacts confidence at all times and among all thinkers who examine it.

By building first on its narrow and exactly conceived foundations and by adding bit by bit to its possessions of permanent truth, mathematics has made possible a release of the imagination of man such as can be completely realized to-day by only a relatively few individuals, a release however which will allow an expanse of the general human mind to-morrow or the next day. Vast new domains of contemplation are opened up by the non-Euclidean geometries, theories of hyperspace and space of an infinite number of dimensions, functions of an infinite number of variables and functions of lines. Such conquests give a new sense of power and mastery and increase the dignity of man. In the presence of so many beautiful creations of his thought "the mathematician lives long and lives young; the wings of his soul do not early drop off;" he rejoices in the grandeur of the heights to which his controlled imagination attains.

If one is to realize the intenser delights afforded by the contemplation of mathematics he must of course be a deep student of its secrets. It is only when he is able to devote a large share of his energy to research and is successful in the creation or discovery of important new truth that he may rejoice in the fullest glow of delight through a realization of himself in such ideal conquests. However important the work of preserving past discoveries and handing down to the future the accumulated tradition and however far-reaching such a stream of influence flowing in hidden ways in the minds of cultivated people, it can not be placed in the same category with that creative work which guides instructor and student alike and teaches generations what to think. It is the great glory of mathematics in our time that its achievements are being immensely enriched and extended by the researches of the present; so that this, the oldest of the sciences, has the vigor and the spring and the growth of the youngest of them. He who discovers a fact or makes known a new law or adds a novel beauty to truth in any way makes every one of us his debtor. How beautiful upon the highway are the feet of him who comes bringing in his hands the gift of a new truth to mankind!

> Alone before the wild and restless force
> Of nature we have seen man's active soul
> Stand forth in awe without a sure resource
> Of power to overcome or to control
> The salient things submerged beneath the whole.
> And we have seen in vision some new power
> Spring up from hidden depths of mind and roll
> With bounding joy to conquest, hour by hour
> Increasing till the strength of man reached fullest flower.

What stage of progress have we now attained
In this process of far-unfolding thought?
What ground to think that it shall be maintained?
Have we the fullness of our conquest brought
And reached the depths where nature works and caught
From her the deepest blessing she can yield?
Or fathomed her profounder secrets fraught
With good, no major truth remaining sealed
From sight, with only minor things to be revealed?

If so, no glow of zeal could move our thought;
Our life would lose its meaning and its zest;
No vision of the future could be caught
By mind's prophetic penetration, blest
With prospect large; in pessimistic rest
And deep stagnation then must mind abide
Without a great compelling interest
To bring its power to action and to guide
Its strength to ways of joy or largest use provide.

But crescent science such a view as this
Dispels; for largest things with keen insight
We feel; the growth of knowledge must dismiss
From thought such pessimism; its darkening blight
Of shadow is illumed by sure foresight.
We joy to see new worth to be attained
And know the present conquest is but slight
Compared with wider truth that shall be gained
For thought's dominions, now by science unexplained.

We need the willing mind to consecrate
Its strength to finding truth, the zeal to bring
From nature's storehouse values good and great
And lay them at the feet of man. To wring
From restive nature some unwilling thing
Were joy supreme. The means for our release
To greater power we seek. The bounding spring
To growth shall move in us and never cease
To bring to us new joy and truth's renowned increase.